LIQUID FLOW BETWEEN ROTATING COAXIAL CYLINDERS TAKING INTO ACCOUNT FRICTIONAL HEATING AND THE DEPENDENCE OF VISCOSITY ON TEMPERATURE.

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A study has been made of the steady motion of a viscous incompressible liquid between two infinite cylinders taking into account frictional heating and the dependence $1/\mu = \varphi(T)$ of viscosity on temperature.

Consider the steady flow of a viscous incompressible liquid between two circular coaxial cylinders of radius r_1 and r_2 ($r_1 < r_2$). For simplicity it is assumed that the inner cylinder is at rest, while the outer one rotates with angular velocity ω_2. (Assuming that the inner cylinder also rotates does not affect the principle.) The temperature of the inner cylinder is T_1, and that of the outer T_2. The viscosity μ of the liquid is assumed to have the following temperature dependence:

$$1/\mu = \varphi(T), \tag{1}$$

where $\varphi(T)$ is a function defined as continuous and monotonically increasing on the interval $[a, +\infty]$, $a \geq 0$, corresponding to all possible values of the temperature.

It is known from [1] that the equations of motion and temperature distribution have the form

$$\frac{d}{dr}(r^2\tau) = 0, \tag{2}$$

$$\frac{\tau^2}{\mu E} + \frac{\lambda}{r}\frac{d}{dr}\left(r\frac{dT}{dr}\right) = 0, \tag{3}$$

where

$$\tau = \mu\left(\frac{du}{dr} - \frac{u}{r}\right) = \mu r\frac{d}{dr}\left(\frac{u}{r}\right). \tag{4}$$

From (2) we have

$$r^2\tau = c_1, \quad \tau = c_1/r^2, \tag{5}$$

and from (3)

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \frac{c_1^2}{\lambda\mu}\frac{1}{r^4} = 0, \tag{6}$$

whence, using (1), we obtain

$$\frac{d^2T}{dr^2} + \frac{1}{r}\frac{dT}{dr} + \nu\frac{1}{r^4}\varphi(T) = 0; \quad \nu = \frac{c_1^2}{\lambda E} \tag{7}$$

with boundary conditions

$$T|_{r=r_1} = T_1, \; T|_{r=r_2} = \tag{8}$$

From (4) and (5) we have

$$\mu r \frac{d}{dr}\left(\frac{u}{r}\right) = \frac{c_1}{r^2}, \tag{9}$$

$$\frac{d}{dr}\left(\frac{u}{r}\right) = \frac{c_1}{\mu r^3} = \frac{c_1 \varphi(T)}{r^3}, \tag{10}$$

$$u|_{r=r_1} = 0, \; u|_{r=r_2} = \omega_2 r_2, \tag{11}$$

whence

$$u = c_1 r \int_{r_1}^{r} \frac{\varphi[T]}{r^3} dr. \tag{12}$$

In (12) T is assumed to be a known function of r and c_1, i. e., $T(r, c_1)$ is obtained by solving (7) for conditions (8). To determine the constant c_1 we then have the equation

$$u_2 = c_1 r_2 \int_{r_1}^{r_2} \frac{\varphi[T(r, c_1)]}{r^3} dr. \tag{13}$$

Instead of the boundary problem (7), (8), consider the integral equation

$$T = \nu \int_{r_1}^{r_2} K(r, \xi) \frac{1}{\xi^4} \varphi[T(\xi)]\, d\xi + \frac{T_2 - T_1}{\ln(r_2/r_1)} + T_1, \tag{14}$$

where $K(r, \xi)$ is a function of the Green operator $\frac{d}{dr}\left(r \frac{dT}{dr}\right)$ with boundary conditions $T|_{r=r_1} = 0$, $T|_{r=r_2} = 0$. The integral equation obtained is investigated by a method similar to that used in [2] and [3].

We state the corresponding results.

1) $\lim\limits_{T \to \infty} \frac{\varphi(T)}{T} = 0$. In this case (14) has a solution for all values of the parameter ν. If we denote the maximum value of the temperature in the flow by T_m, then, as $\nu \to \infty$, $T_m \to \infty$ and, as $\nu \to 0$, the solution of (14) tends to the solution corresponding to motionless cylinders. For sufficiently small ν the solution is unique.

2) $\lim\limits_{T \to \infty} \frac{\varphi(T)}{T} = B$, $0 < B < \infty$. In this case a value ν_0 exists such that for $\nu < \nu_0$ (14) has a solution, while for $\nu > \nu_0$ there is no solution. Further, there is a value $\nu_1 \leq \nu_0$ such that the equation has a solution for which $T_m \to \infty$ as $\nu \to \nu_1$. If $\nu_1 < \nu_0$, then for $\nu_1 < \nu < \nu_0$ the equation has at least two solutions. For sufficiently small ν the solution is unique and, as $\nu \to 0$, the solution tends to that corresponding to motionless cylinders.

3) $\lim\limits_{T \to \infty} \frac{\varphi(T)}{T} = \infty$. In this case, too, there is a critical value ν_0; for $\nu < \nu_0$ the equation has a solution, and for $\nu > \nu_0$ there is no solution. Then for $0 < \nu < \nu_0$ the equation has at least two solutions. As $\nu \to 0$, one of these tends to the solution corresponding to motionless cylinders, while for the other, as $\nu \to 0$, $T_m \to \infty$.

Since $\nu = c_1^2/\lambda E$, and $\tau = c_1/r^2$, $\tau = \sqrt{\nu \lambda E}/r^2$. Thus, in Case 1 flow regimes are possible with arbitrarily large internal friction stresses, while in Cases 2 and 3 the possible values of the internal friction have upper limits.

Note that in Cases 2 and 3 the maximum permissible value of the internal friction stress depends on r_1, r_2 and T_1, T_2, on the form of function $\varphi(T)$, and on the physical properties of the liquid, but does not depend on the angular velocity of the cylinder.

Let us now investigate possible flow regimes as a function of the speed of the outer cylinder. Equation (14) may have more than one solution for a given value of ν, but on each branch the solution is a continuous function of ν (and consequently of c_1). Let us examine the three cases enumerated separately.

1) In this case, as c_1 increases from 0 to ∞, $T(r_1 c_1)$ also increases to ∞. It then follows from (13) that u_2 also varies from 0 to ∞. Thus, in this case as $u_2 \to \infty$ the internal friction stress increases without limit. It is easy to see that for any value of u_2 Eq. (13) may be solved with respect to c_1.

2) In this case, as c_1 increases from 0 to a value corresponding to the critical value ν_0, $T_m \to \infty$ for one of the solutions, so that from (13) we have $u_2 \to \infty$. When $c_1 \to 0$, $u_2 \to 0$. The following flows are possible: a) as u_2 increases from 0 to ∞, the internal friction stress increases, approaches a maximum, but does not reach it (then $\nu_1 = \nu_0$); b) as u_2 increases from 0 to ∞, the internal friction stress first grows, attains a maximum value, and then begins to decrease, approaching a value other than zero (in this case $\nu_1 < \nu_0$). It is clear that (13) may be solved with respect to c_1 for any u_2.

3) It is not hard to establish that in this case, as u_2 increases from 0 to ∞, the internal friction stress increases, attains a maximum for some finite value of u_2, and then, with further increase, begins to decline, tending to zero. In this case (13) can always be solved with respect to c_1.

It has thus been shown that steady flow exists for any boundary values of temperature and velocity, and for any kind of dependence of viscosity on temperature.

It is interesting to compare the result with that of [2]. In [2] Kaganov dealt with the problem of motion of a liquid in a two-dimensional channel and in an infinite circular cylindrical tube under the influence of a pressure gradient, allowing for frictional heating and variation of viscosity with temperature.

He established that in Cases 2 and 3 it is impossible to have values of ν greater than the critical value for steady flow regimes.

Note, finally, that the problem investigated has been studied in [4] and [5] for particular forms of the dependence of viscosity on temperature. In [4] the function $\varphi(T)$ takes the form $\varphi(T) = aT + b$, and in [5] the form $\varphi(T) = A\exp \delta T$. In these cases the corresponding equation is integrated in known functions, and the solution may be investigated directly. The authors of [4] obtain a result that is a corollary of our Case 2. The case $\varphi(T) = A\exp\delta T$ is our special Case 3. The author of [5], however, did not establish the characteristic boundedness of the internal friction. Moreover, discarding the second, physically real solution of (7), he reaches the erroneous conclusion that a steady regime is impossible if his condition (28) is not satisfied. It is easy to show that condition (28) of [5] does not make sense physically. Actually, this condition has the form

$$T_2 - T_1 < \frac{2}{\delta} \ln \frac{r_2}{r_1}$$

and, since it does not contain the hydrodynamic parameters of motion, cannot be used to obtain a criterion of impossibility of steady flow.

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